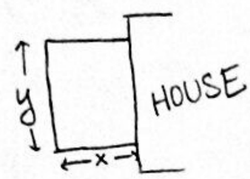
6. You do more reading and discover that the optimal size of an exercise pen is 40 ft^2. What is the minimum amount of fence you can use to create the pen, if you're still bounding one side of the pen with the wall of your house?

① 

② Minimize $P = 2x + y$

③ $x, y > 0$

④ Constraint: $xy = 40$

Solve for y: $y = \frac{40}{x}$

Rewrite P: $P = 2x + \frac{40}{x}$

⑤ $P' = 2 - \frac{40}{x^2}$

$= \frac{2x^2 - 40}{x^2}$

CV's: $2x^2 - 40 = 0$, $x^2 = 0$

$x = \sqrt{20}$, $\cancel{x = -\sqrt{20}}$, $\cancel{x = 0}$

1st Derivative Test:

P': $-$ $+$

x: 0 $\sqrt{20}$

absolute min

⑥ $x = \sqrt{20} = \sqrt{2^2 \cdot 5} = 2\sqrt{5}$

$P = 2(2\sqrt{5}) + \frac{40}{2\sqrt{5}}$

$= 4\sqrt{5} + \frac{20}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = 4\sqrt{5} + 4\sqrt{5}$

$= \boxed{8\sqrt{5} \text{ ft}}$

7. Coke needs to make a cylindrical can that can hold precisely 500 cm^3 of liquid. Find the dimensions of the can that will minimize the cost.

The volume of a cylinder is $\pi r^2 h$ and the surface area is $2\pi rh + 2\pi r^2$ where r is the radius and h is the height.

① 

② Minimize $A = 2\pi rh + 2\pi r^2$

③ $r, h > 0$

④ Constraint: $\pi r^2 h = 500$

Solve for h: $h = \frac{500}{\pi r^2}$

Rewrite A: $A = 2\cancel{\pi}\cancel{r}\left(\frac{500}{\cancel{\pi} r^{\cancel{2}}}\right) + 2\pi r^2$

$= \frac{1000}{r} + 2\pi r^2$

⑤ $A' = \frac{-1000}{r^2} + 4\pi r$

$= \frac{-1000 + 4\pi r^3}{r^2}$

CV's: $-1000 + 4\pi r^3 = 0$, $r^2 = 0$

$4\pi r^3 = 1000$ $\quad \cancel{r = 0}$

$r^3 = \frac{\cancel{1000}\ 250}{\cancel{4}\pi}$

$r = \sqrt[3]{250/\pi}$

1st Derivative Test:

A': $-$ $+$

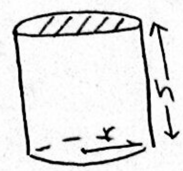
r: 0 $\sqrt[3]{250/\pi}$

⑥ $r = \sqrt[3]{\frac{250}{\pi}} = \sqrt[3]{\frac{5^3 \cdot 2}{\pi}} = \boxed{5\sqrt[3]{2/\pi} \text{ cm}}$

$h = \frac{500}{\pi(5\sqrt[3]{2/\pi})^2} = \frac{500}{25\pi \sqrt[3]{4}\, \pi^{-2/3}}$

$= \boxed{\frac{20}{\sqrt[3]{4\pi}} \text{ cm}}$

8. Ted wants to construct a cylindrical flower pot. He has 60ft^2 of metal. What is the volume of the largest flower pot he can build? Recall that the volume of a cylinder is $\pi r^2 h$ and the surface area of a cylinder with open top is $\pi r^2 + 2\pi rh$ where r is the radius and h is the height.

① 

② Maximize $V = \pi r^2 h$

③ $r, h > 0$

④ Constraint: $\pi r^2 + 2\pi r h = 60$

Solve for h: $2\pi r h = 60 - \pi r^2$

$h = \dfrac{60 - \pi r^2}{2\pi r}$

Rewrite V: $V = \cancel{\pi} r^{\cancel{2}}\left(\dfrac{60-\pi r^2}{2\cancel{\pi}\cancel{r}}\right)$

$= r\left(\dfrac{60-\pi r^2}{2}\right) = 30r - \frac{\pi}{2} r^3$

⑤ $V' = 30 - \frac{3\pi}{2} r^2 \overset{set}{=} 0$

CVs: $\frac{3\pi}{2} r^2 = 30$

$r^2 = \overset{10}{\cancel{30}}\left(\frac{2}{\cancel{3}\pi}\right) = \frac{20}{\pi}$

$r = \sqrt{20/\pi}$, $\cancel{r = -\sqrt{20/\pi}}$

2nd Derivative Test:

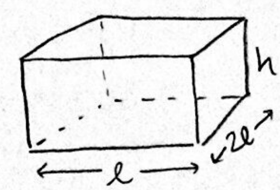
$V'' = -3\pi r < 0$ when $r = \sqrt{20/\pi}$

⑥ $r = \sqrt{20/\pi} = \sqrt{\frac{2^2 \cdot 5}{\pi}} = 2\sqrt{5/\pi}$

$V = 30\left(2\sqrt{\frac{5}{\pi}}\right) - \frac{\pi}{2}\left(2\sqrt{5/\pi}\right)^3$

$\approx \boxed{50.46 \text{ ft}^3}$

9. A moving box with width twice the length needs to have a volume of 32ft^3. Find the smallest possible amount of material needed.

① 

② Minimize $A = 2(2l^2) + 2(2lh) + 2(lh)$

$= 4l^2 + 4lh + 2lh$

$= 4l^2 + 6lh$

③ $l, h > 0$

④ Constraint: $l(2l)(h) = 32$

Solve for h: $h = \dfrac{32}{2l^2} = \dfrac{16}{l^2}$

Rewrite A: $A = 4l^2 + 6\cancel{l}\left(\dfrac{16}{l^{\cancel{2}}}\right)$

$= 4l^2 + \dfrac{96}{l}$

⑤ $A' = 8l - \dfrac{96}{l^2}$

$= \dfrac{8l^3 - 96}{l^2}$

CV's: $8l^3 - 96 = 0$, $l^2 = 0$

$8l^3 = 96$

$l^3 = 12$

$l = \sqrt[3]{12}$, $\cancel{l = 0}$

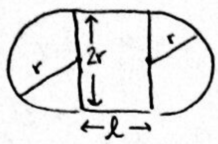
2nd Derivative Test:

$A'' = 8 + \frac{192}{l^3} > 0$ when $l = \sqrt[3]{12}$

⑥ $A = 4(\sqrt[3]{12})^2 + \dfrac{96}{\sqrt[3]{12}}$

$\approx \boxed{62.898 \text{ ft}^2}$

10. Meriel wants to construct a circular table with a rectangular leaf that can be added in the middle. She wants the perimeter of the table (with the leaf in) to be 12 feet. What dimensions maximize the size of the leaf?

① 

② Maximize $A = 2r\ell$

③ $r, \ell > 0$

④ Constraint: $2\ell + 2\pi r = 12$

Solve for ℓ: $2\ell = 12 - 2\pi r$

$\ell = 6 - \pi r$

Rewrite A: $A = 2r(6 - \pi r)$

$= 12r - 2\pi r^2$

⑤ $A' = 12 - 4\pi r \overset{\text{set}}{=} 0$

$4\pi r = 12$

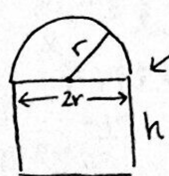
CV: $r = \frac{12}{4\pi} = \frac{3}{\pi}$

2nd Derivative Test:

$A'' = -4\pi < 0$ ∩

⑥ $r = \boxed{\frac{3}{\pi}\text{ ft}}$, $\ell = 6 - \pi\left(\frac{3}{\pi}\right) = \boxed{3\text{ ft}}$

11. Vincent plans to build a dog house whose cross section is a rectangle surmounted by a semicircle. If he wants the girth (perimeter of the cross section) to be 16 feet, what is the largest cross sectional area possible?

① 

Cross Section

② Maximize $A = A_{rect} + A_{semicircle}$

$= 2rh + \frac{1}{2}\pi r^2$

③ $r, h > 0$

④ Constraint: $2r + 2h + \frac{1}{2}(2\pi r) = 16$

Solve for h: $2h = 16 - 2r - \pi r$

$h = 8 - r - \frac{\pi}{2}r$

Rewrite A: $A = 2r\left(8 - r - \frac{\pi}{2}r\right) + \frac{1}{2}\pi r^2$

$= 16r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$

$= 16r + \left(-2 - \pi + \frac{1}{2}\pi\right)r^2$

$= 16r + \left(-2 - \frac{1}{2}\pi\right)r^2$

⑤ $A' = 16 + 2\left(-2 - \frac{1}{2}\pi\right)r$

$= 16 + (-4 - \pi)r \overset{\text{set}}{=} 0$

$(-4 - \pi)r = -16$

$r = \frac{-16}{-4-\pi} \cdot \frac{-1}{-1} = \frac{16}{4+\pi}$

2nd Derivative Test:

$A'' = 2\left(-2 - \frac{1}{2}\pi\right) < 0$ ∩

⑥ $A = 16\left(\frac{16}{4+\pi}\right) + \left(-2 - \frac{1}{2}\pi\right)\left(\frac{16}{4+\pi}\right)^2$

$\approx \boxed{17.92\text{ ft}^2}$